Simplex

Systematic approach to solve linear programming

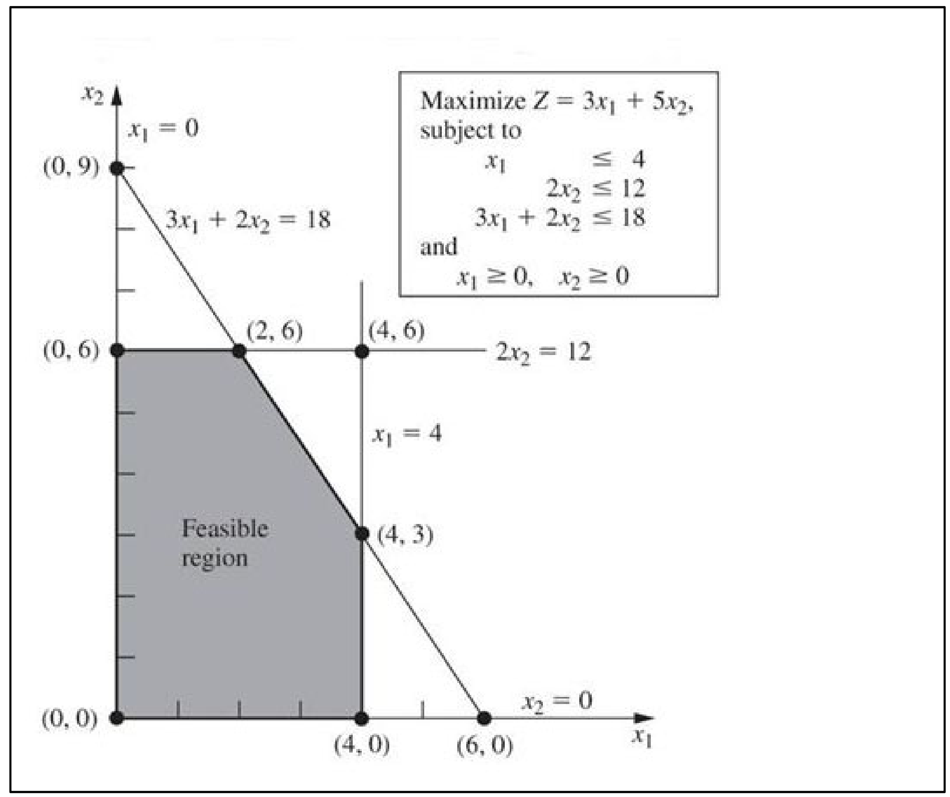
Simplex

- Optimum solutions are always associated with corner points of the solution space .
- The simplex method always starts at a feasible corner point of the feasible region, and always passes through an adjacent feasible corner point, checking each point for optimality before moving to a new one.
- Basically, this method gives a systematic way of moving from one corner to another one in the feasible region in such a way that the value of the objective function increases until an optimum value is reached or it is discovered that no solution exists.

Credit: Prof. Elbeltagi

Two variable simplex search

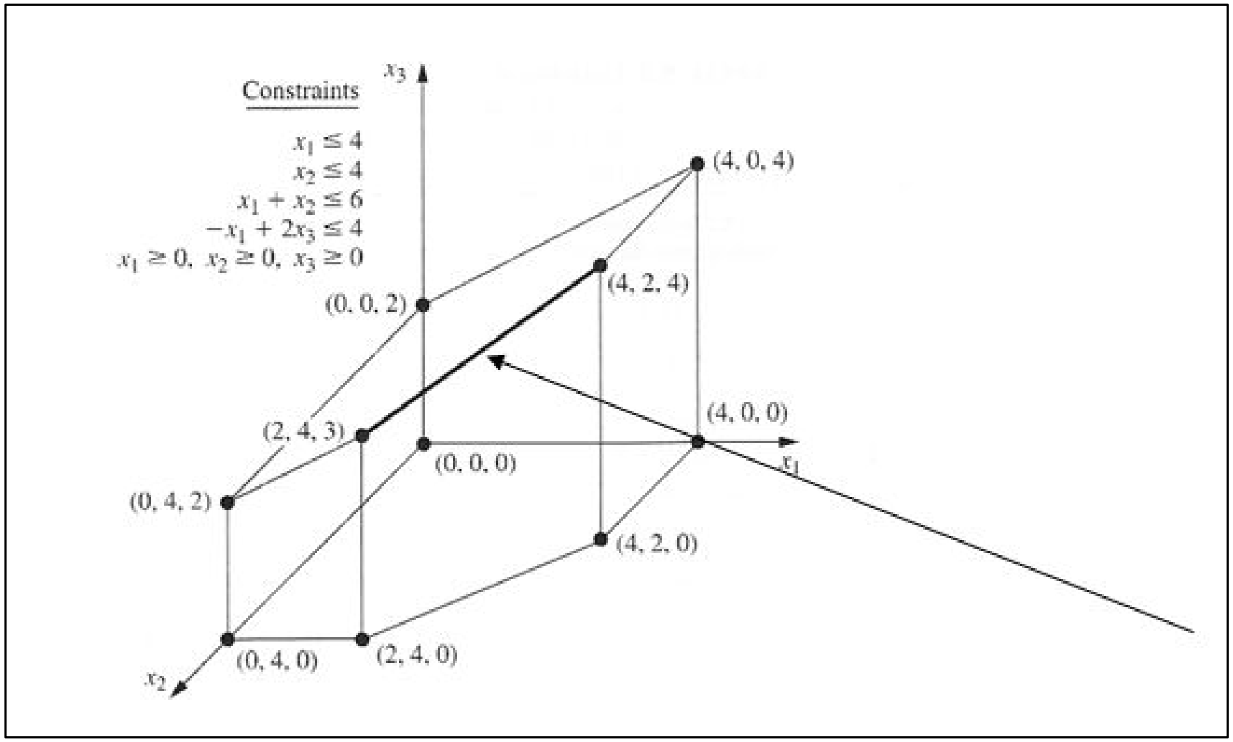
- Corner-Point Feasible (CPF)
- Constraints and CPFs
- Adjacent CPFs solutions



Three variables

- Algebraically, we know that each of the CPFs for n variables, there are n constraints that create this CPF.

- Moving to an adjacent CPF requires moving across one of the constraints.



Simplex Method

- We can generalize any linear programming problem to the following format to deal with any number of variables:

$$
\begin{aligned}
&\textit{Max} && c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n \\
&\textit{subject to} && a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& && a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \\
& && x_1 \geq 0;\ \ldots\ldots\ x_n \geq 0
\end{aligned}
$$

Simplex Method

- Slack variables and solution search

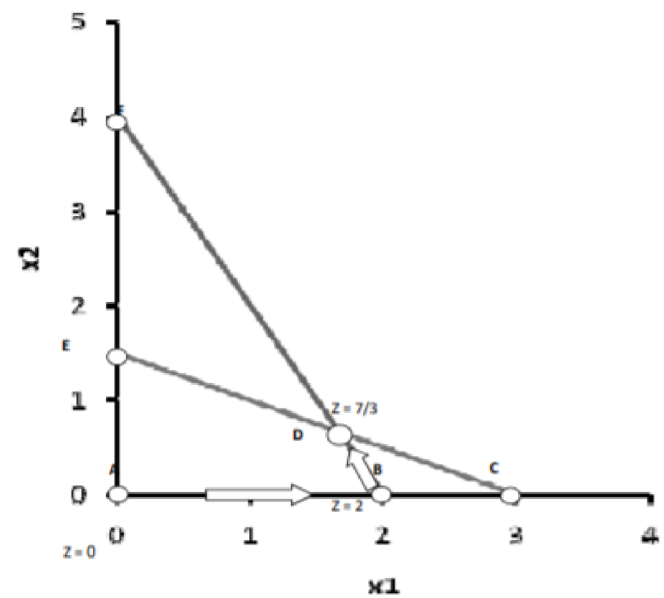
Min	$-2x_1 + 3x_2$		*Max*	$2x_1 - 3x_2$
	$x_1 - 3x_2 + 2x_3 \leq 3$			$x_1 - 3x_2 + 2x_3 + s_1 = 3$
	$-x_1 + 2x_2 \geq 2$			$-x_1 + 2x_2 - s_2 = 2$

Simplex Method

- Basic and non-Basic variables (s_1,s_2)

$$
\begin{aligned}
z - \quad x_1 - \quad x_2 \quad & & &= 0\\
2x_1 + \quad x_2 \quad + s_1 & & &= 4\\
x_1 + \quad 2x_2 \quad & + s_2 & &= 3
\end{aligned}
$$

$$x_1 = x_2 = 0;\ s_1 = 4;\ s_2 = 3;\ z = 0$$



Simplex Method Steps

- First, use augmented format
- Use tabular form
- Per each step, determine the entering and leaving variables
- Repeat until optimal solution is reached (no negative coefficients in objective function).

Example

Max $Z = x_1 + x_2$

St.

$2x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

- First, use augmented format
- Use tabular form

$$
\begin{aligned}
z - \quad x_1 - \quad x_2 \qquad &= 0 \\
2x_1 + \quad x_2 + s_1 \qquad &= 4 \\
x_1 + 2x_2 \qquad + s_2 \quad &= 3
\end{aligned}
$$

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	-1	-1	0	0	0
s_1	0	2	1	1	0	4; 4/2=2
s_2	0	1	2	0	1	3; 3/1=3

Basic variables s_1 =4 and s_2 = 3
No-basic variables x_1, x_2 = 0

Example

Max Z = $x_1 + x_2$

St.

$2x_1 + x_2 \le 4$

$x_1 + 2x_2 \le 3$

$x_1, x_2 \ge 0$

- Per each step, determine the entering and leaving variables
  - Entering variable (maximum negative coefficient in Z)
  - Leaving variable (smallest non-negative ratio; RHS/coefficient of entering variable)

$$
\begin{aligned}
z - \quad x_1 - x_2 \quad &= 0 \\
2x_1 + x_2 + s_1 \quad &= 4 \\
x_1 + 2x_2 \quad + s_2 &= 3
\end{aligned}
$$

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	-1	-1	0	0	0
s_1	0	2	1	1	0	4; 4/2=2
s_2	0	1	2	0	1	3; 3/1=3

Basic variables $s_1 = 4$ and $s_2 = 3$

No-basic variables $x_1, x_2 = 0$

Example

Max $Z = x_1 + x_2$

St.

$2x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

- New table:
  1. New pivot row = old row / pivot element
  2. New row = old row – (pivot column coefficient * new pivot row coefficient)

Pivot Element

Pivot Colum

New Pivot Row

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	-1	-1	0	0	0
s_1	0	2	1	1	0	4
s_2	0	1	2	0	1	3
z	1	0	-1/2	1/2	0	2
x_1	0	1	½	½	0	2; 2/(1/2) = 4
s_2	0	0	3/2	-1/2	1	1; 1/(3/2) = 2/3

Basic variables x_1 =2 and s_2 = 1

No-basic variables s_1, x_2 = 0

Example

Max $Z = x_1 + x_2$

St.

$2x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

- Repeat till no negative coefficient in Z equation

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	-1	-1	0	0	0
s_1	0	2	1	1	0	4
s_2	0	1	2	0	1	3
z	1	0	-1/2	1/2	0	2
x_1	0	1	½	½	0	2
s_2	0	0	3/2	-1/2	1	1
z	1	0	0	1/3	1/3	7/3
x_1	0	1	0	2/3	-1/3	5/3
x_2	0	0	1	-1/3	2/3	2/3

Basic variables x_1 =5/3 and x_2 = 2/3

No-basic variables s_1, s_2 = 0

Example – Building Houses

- Decision Variables
  - x_1 Number of house type A
  - x_2 Number of houses type B
  - x_3 Number of houses type C

- Objective: Maximize profit(in 1000s): $Z = 100x_1 + 300x_2 + 200x_3$

- Constraints:
  - Total houses built $x_1 + x_2 + x_3 \leq 100$
  - Needed resources $40x_1 + 20x_2 + 30x_3 \leq 3200$
  - Needed engineers $x_1 + 2x_2 + x_3 \leq 160$
  - Non-negative $x_1, x_2, x_3 \geq 0$

Example – Building Houses

- Standard form

$$Z - 100\,x_1 - 300\,x_2 - 200\,x_3 \qquad = 0$$

$$x_1 + \quad x_2 + \quad x_3 + s_1 \quad = 100$$

$$40\,x_1 + 20\,x_2 + 30\,x_3 \quad + s_2 \quad = 3200$$

$$x_1 + 2\,x_2 + \quad x_3 \quad + s_3 = 160$$

$$x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0;\ s_1 \geq 0;\ s_2 \geq 0;\ s_3 \geq 0$$

Steps

- ✓ First, use augmented format
- Use tabular form
- Per each step, determine the entering and leaving variables
- Repeat until optimal solution is reached (no negative coefficients in objective function).

Example – Building Houses

Basic Variables	Coefficient of							R.H.S	Ratio	
	Z	x1	x2	x3	s1	s2	s3			
Z	1	-100	-300	-200	0	0	0	0		R0 (objective)
s1	0	1	1	1	1	0	0	100	100 / 1 = 100	R1 (constraint 1)
s2	0	40	20	30	0	1	0	3200	3200 / 20 = 160	R2 (constraint 2)
s3	0	1	2	1	0	0	1	160	160 / 2 = 80	R3 (constraint 3)
R3n	0	0.5	1	0.5	0	0	0.5	80		

New pivot row = old row / pivot element

Steps

- ✓ First, use augmented format
- ✓ Use tabular form
- ✓ Per each step, determine the entering and leaving variables
- Repeat until optimal solution is reached (no negative coefficients in objective function).

Example – Building Houses

Basic Variables	Coefficient of							R.H.S	Ratio	
	Z	x1	x2	x3	s1	s2	s3			
Z	1	50	0	-50	0	0	150	24000		R0n = (R0+300R3n)
s1	0	0.5	0	0.5	1	0	-0.5	20	20 / 0.5 = 40	R1n = (R1-R3n)
s2	0	30	0	20	0	1	-10	1600	1600 / 20 = 80	R2n = (R2-20R3n)
x2	0	0.5	1	0.5	0	0	0.5	80	80 / 0.5 = 160	R3n = (0.5R3)
R1n	0	1	0	1	2	0	-1	40		

New row = old row – (pivot column coefficient * new pivot row coefficient)

Steps

- ✓ First, use augmented format
- ✓ Use tabular form
- ✓ Per each step, determine the entering and leaving variables
- Repeat until optimal solution is reached (no negative coefficients in objective function).

Example – Building Houses

Basic Variables	Coefficient of							R.H.S	
	Z	x1	x2	x3	s1	s2	s3		
Z	1	100	0	0	100	0	100	26000	R0 = (R0n+50R1)
x3	0	1	0	1	2	0	-1	40	R1 = (2R1n)
s2	0	10	0	0	-40	1	10	800	R2 = (R2n-20R1)
x2	0	0	1	0	-1	0	1	60	R3 = (R3n-0.5R1)

Steps

- ✓ First, use augmented format
- ✓ Use tabular form
- ✓ Per each step, determine the entering and leaving variables
- ✓ Repeat until optimal solution is reached (no negative coefficients in objective function).

Special Cases in Simplex

- If you have ***tie in selecting entering variables***, you may choose any of them.

- You may find ***alternate optimal solution***. After you are done, you will find a non-basic variable that has a zero coefficient in the objective function.

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	**-1**	-1/2	0	0	0
s_1	0	**2**	1	1	0	**4**
s_2	0	**1**	2	0	1	3
z	1	0	**0**	½	0	2
x_1	0	1	½	½	0	2; 2/(1/2) = 4
s_2	0	0	**3/2**	-1/2	1	**1; 1/(3/2) = 2/3**

Special Cases in Simplex

- ***Unbounded solution, no leaving variable,*** will be present when all the coefficient of leaving variables are negative.

Basic variables	Coefficient of					Right-hand side (solution)
	z	x_1	x_2	s_1	s_2	
z	1	**-2**	-1	0	0	0
s_1	0	-1	1	1	0	1
s_2	0	1	-2	0	1	**2; 2/1 = 2**
z	1	0	**-5**	0	2	4
s_1	0	0	**-1**	1	1	3
x_1	0	1	**-2**	0	1	2

- ***Infeasible solution***, will be present when you achieve a negative value for a basic variable

Special Cases in Simplex

- Greater than or equal (≥) Constraints

$$\begin{aligned} &Max \quad 2x_1 - 3x_2 \\ &x_1 - 3x_2 + 2x_3 + s_1 = 3 \\ &- x_1 + 2x_2 - s_2 = 2 \end{aligned}$$

Thus s_2 is a basic variable with value of **-2**

- Equality (=) Constraints

$3x_1 + 2x_2 = 18$

$3x_1 + 2x_2 + s1 = 18$

Thus we created a new basic variable that *must be equal zero*.

Introducing the Big M Concept

- Consider the Big M as a number almost as big as infinity (but not infinity).
- We can manipulate our equations to create a new non-basic variable. It can be multiplied by M, as it will always generate a zero value.
- The trick is to know how and when to use the new non-basic variable, and when to multiple it by M.

Solving for Equality

Maximize $3x_1 + 5x_2$

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 = 18$

$x_1 \geq 0;\ x_2 \geq 0$

The augmented form becomes

$$z - 3x_1 - 5x_2 = 0$$

$$x_1 + s_1 = 4$$

$$2x_2 + s_2 = 12$$

$$3x_1 + 2x_2 = 18 \Rightarrow 3x_1 + 2x_2 + s_3 = 18$$

$$\Uparrow$$

$$Z - 3x_1 - 5x_2 + MS_3 = 0$$

Without this, our math incorrect, cause the last equation then states that 0 = 18

<u>But even then we still have the same problem</u>

Solving for Equality

- We can manipulate the math to create a new Z function that allows s3 to be basic.

$$Z - 3x_1 - 5x_2 + MS_3 = 0$$

$$3x_1 + 2x_2 + s_3 = 18 \qquad \text{X (-M)}$$

$$=$$

$$z - (3 + 3\text{M})x_1 - (5 + 2\text{M})x_2 = -18\text{M}$$

Solving for Greater than or Equal (≥)

Minimize $Z = 0.4x_1 + 0.5x_2$

Subject to

$$0.3x_1 + 0.1x_2 \le 2.7$$
$$0.5x_1 + 0.5x_2 = 6$$
$$0.6x_1 + 0.4x_2 \ge 6$$

$x_1 \ge 0;\ x_2 \ge 0$

$$0.6x_1 + 0.4x_2 \ge 6$$

Change to: $0.6x_1 + 0.4x_2 - s_3 = 6 \quad (s_3 \ge 0)$

Then: $0.6x_1 + 0.4x_2 - s_3 + s_4 = 6 \quad (s_3 \ge 0,\ s_4 \ge 0)$

S_3 = -6 ??

Add S_4 but add it to Z function with a multiplication by M

Solving for Greater than or Equal (≥)

Maximize $-Z + 0.4x_1 + 0.5x_2 + M\,s_2 + M\,s_4 = 0$

Subject to

$0.3\,x_1 + 0.1\,x_2 + s_1 = 2.7$

$0.5\,x_1 + 0.5\,x_2 + s_2 = 6$

$0.6\,x_1 + 0.4\,x_2 - s_3 + s_4 = 6$

$x_1 \geq 0;\; x_2 \geq 0;\; s_1 \geq 0;\; s_3 \geq 0;\; s_4 \geq 0.$

Now, we have fixed the minimization to maximization, we have fixed the equality constraint, but still s_3 = -6, and 0 = 6.

We can manipulate our math to recreate the Z function with different variables.

$-Z + 0.4x_1 + 0.5x_2 + M\,s_2 + M\,s_4 = 0$

$0.6\,x_1 + 0.4\,x_2 - s_3 + s_4 = 6$ X (-M)

$0.5\,x_1 + 0.5\,x_2 + s_2 = 6$ X (-M)

=

$-Z + x_1(0.4 - 1.1M) + x_2(0.5 - 0.9M) + M\,s_3 = -12M$

Summary on Big M

- In case of Equality or Greater than, we have to add a non-basic variable.
  - Add it in the equation and in Z function multiplied by M.
  - Manipulate the Z function by subtracting it from the equation multiplied by M.

Continue with the Tabular Form

Basic variables	Coefficient of							Right-hand side
	Z	x_1	x_2	s_1	s_2	s_3	s_4	
Z	-1	-1.1M+0.4	-0.9M+0.5	0	0	M	0	-12M
s_1	0	0.3	0.1	1	0	0	0	2.7
s_2	0	0.5	0.5	0	1	0	0	6
s_4	0	0.6	0.4	0	0	1	1	6

Example

- John has $20,000 to invest in three funds F1, F2 and F3. Fund F1 is offers a return of 2% and has a low risk. Fund F2 offers a return of 4% and has a medium risk. Fund F3 offers a return of 5% but has a high risk. To be on the safe side, John invests no more than $3000 in F3 and at least twice as much as in F1 than in F2. Assuming that the rates hold till the end of the year, what amounts should he invest in each fund in order to maximize the year end return?