Decision Analysis and Game Theory

CB714 – Advanced System Analysis

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What would you do when you face options?

- Which car would you buy, BMW 320i vs. Honda Accord VTI?

- Which elective class would you take?

How do you make a choice?

Utility Theory

- Utility is how you measure your preference over different options
  - For cars: Fast, reliable, economic, etc.

- Utility thus enables us to determine the option that maximizes our objectives.

- Utility requires the player (decision maker) to be rational.

Decision Theory

- If we are faced with multiple options, knowing the utility (outcome) of each decision, we can determine the best alternative for ourselves.

- In decision theory, a player (decision maker) faces nature to maximize his/her own outcome.

$$d \rightarrow \mathrm{ArgMax}\ [u]_{d}$$

d: decision, u: utility

Stochastic Nature

- What if we know only the utility, but with some probability of happening.
- Reflecting on behavior towards *risk*.
- Consider the following example (Payoff table)

decision	State of nature	
	Oil	Dry
Drill for oil	*LE700,000*	*-LE100,000*
Sell the land	*LE90,000*	*LE90,000*
Chance of state	¼	¾

Credit: Prof. Elbeltagi

State of Nature

- Nature is uncontrollable. However, it reacts with some rules, laws, and probabilities.

decision	State of nature	
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Drill for oil	*LE700,000*	*-LE100,000*
Sell the land	*LE90,000*	*LE90,000*
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Expected Utility

- Since the utility of each choice is not deterministic, we need to calculate the Expected Utility, *E[u].*
- Expected utility is the outcome of each decision through the multiplication of the utility (*u*) by the probability of happening (*P*)

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Expected Utility

- $E[u]_{Drill} = 0.25 * 700{,}000 + 0.75 * -100{,}000 =$ LE 100,000
- $E[u]_{Sell} = 0.25 * 90{,}000 + 0.75 * 90{,}000 =$ LE 90,000

decision	State of nature	
	Oil	Dry
Drill for oil *Sell the land*	*LE700,000* *LE90,000*	*-LE100,000* *LE90,000*
Chance of state	¼	¾

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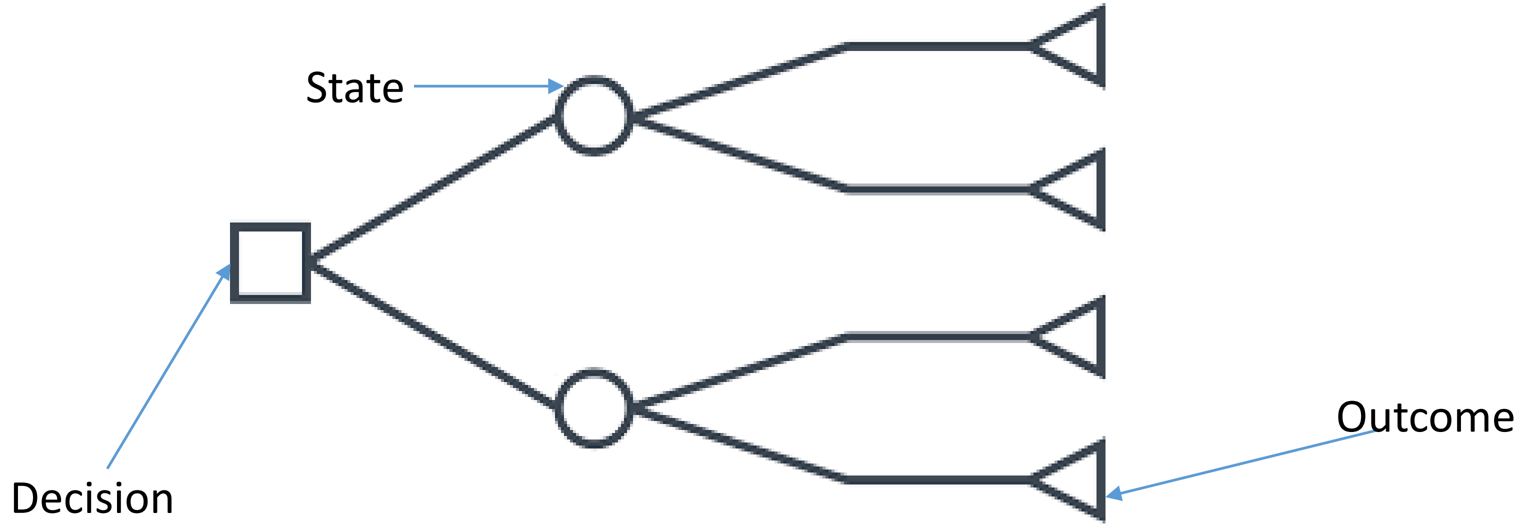
Example - Gambling

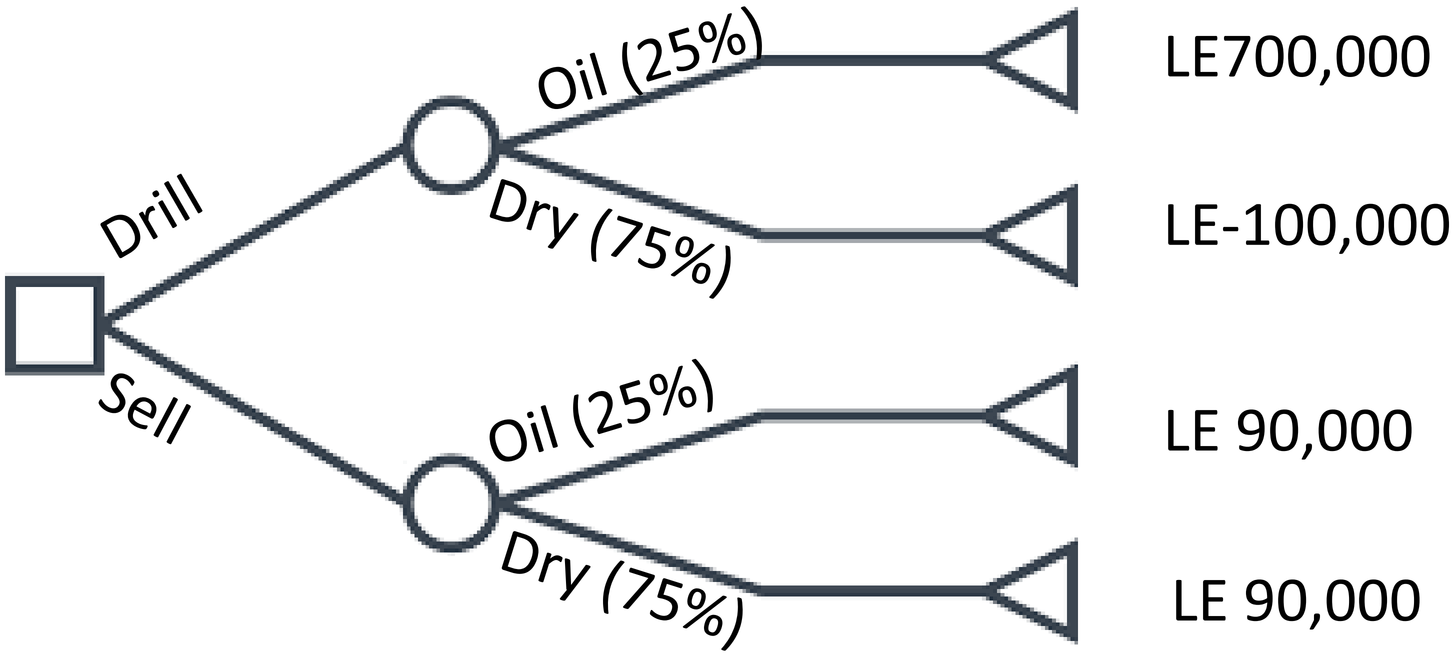
decision	State of nature	
	Win	Loose
Play *Do not play*	*LE6* *LE0*	*-LE9* *LE0*
Chance of state	*2/3*	*1/3*

Decision Tree

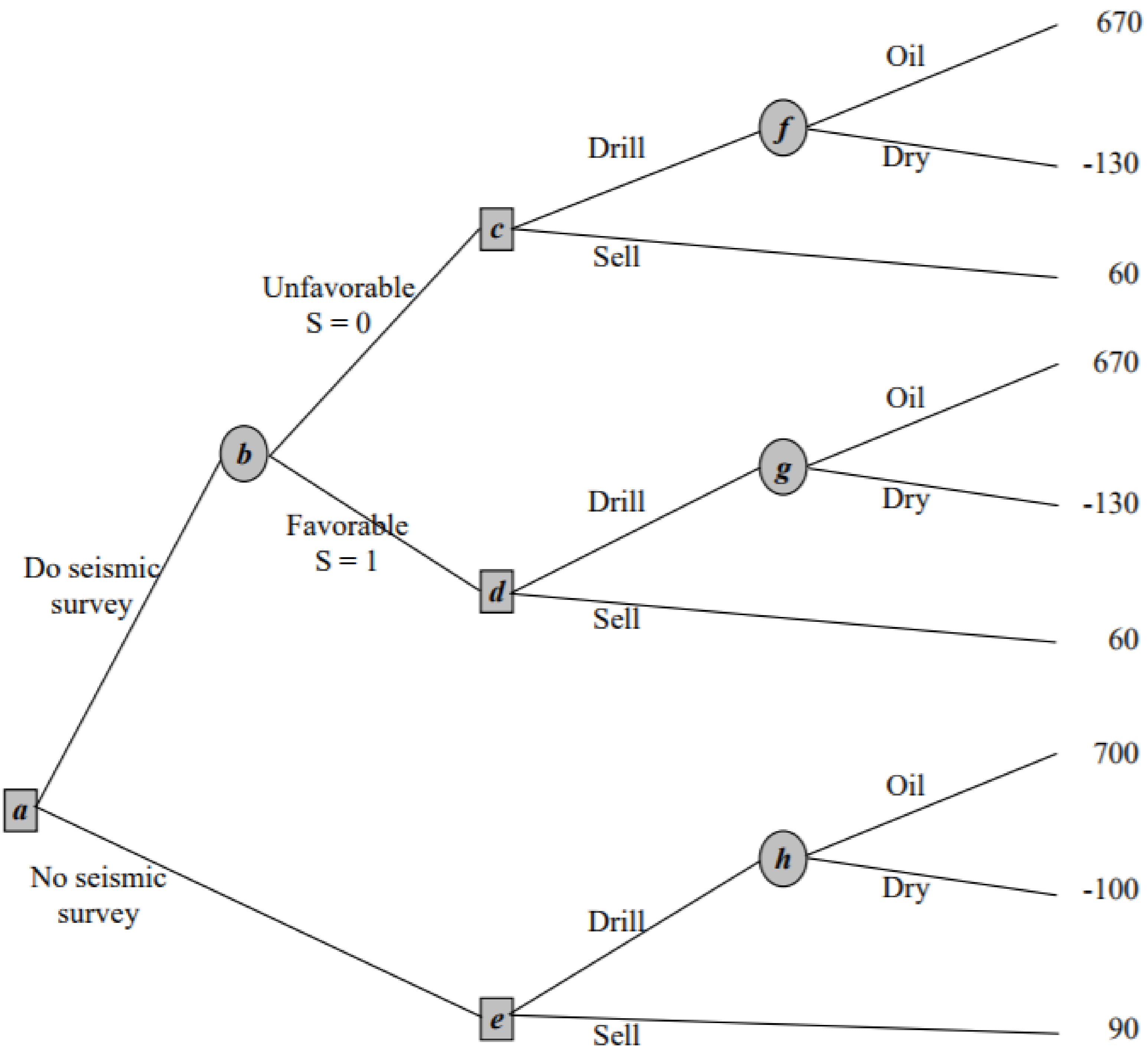
- When facing multiple options, decision tree can be very useful.
- Decision tree is even better in extensive decision (decisions within decisions).



Decision Tree for Oil example



Backward induction: solving from right to left


a
b
c
d
e
f
g
h
Do seismic survey
No seismic survey
Unfavorable S = 0
Favorable S = 1
Drill
Sell
Oil
Dry
670
-130
60
670
-130
60
700
-100
90

New Developing Company

- A new developing company is considering to enter the Egyptian market.
- Currently, there multiple of big firms (for simplicity, let’s consider them as one big competitor).
- The market now has two potentials, 70% to stay normal, otherwise, there will be a recession.
- The new company also knows that if it entered the market, the other companies might increase their prices with 50% probability.
- If the other companies increased their prices, the new company will have 100 units of profit if the market is in normal conditions, and only 40 units profit otherwise.
- if the other companies did not increase their prices, the new company will have 20 units profit if the market is in normal conditions, and -30 units of profit otherwise.

Now lets assume the other players decision

- The older companies will have 100 units of profit if they increased their prices with normal market conditions, and 40 with recession, if a new company entered the market.

- The older companies will have 30 for low prices and normal conditions, and -40 for recession. If a new company entered the market.

- The older companies will have 300 units profit, if they increased the prices with normal market conditions, and 240 with recession, and no one entered the market.

- The older companies will have 50 units of profit if they did not increase the prices, the market is in normal conditions, -10 in recession, and no one entered the market

How does that affect the outcome?

Game Theory

Game Theory

- Games are *systems* where *players* take strategic decision to maximized their *utilities*.
- In game theory, a player does not play against nature, but against another *rational* player.
- Unlike decision theory, game theory does not seek merely optimality for a player, but equilibrium.

Game Theory

Equilibrium vs. Optimality.

Game Time

Prisoners Dilemma

Game Theory

- Game theory is a mathematical formulation of conflict and cooperation based on strategic decision making of rational players

Rules of the Game

- PAPI
  - Players
  - Actions
  - Payoffs
  - Information

Players (*i*)

Players are rational individuals who decide on actions to maximize their utilities.

For simplicity, a game of *n* players, we will denote the player as *i*, and the other players as -*i*

Actions

- An action by player (*i) is denoted as* a_i
- A player can thus have a set of available actions $A_i = \{a_i\}$
- Action combination of players $a = \{a_i\}\ \forall i \ \in n$
- Do NOT get confused between <u>*actions*</u> and <u>*strategies*</u>

Payoffs

- Payoffs π_i can be explained as
  - the utility the player *i* receives after he/she and the others players picked their strategies and the game has been played
  - The expected utility as a function of the utilized strategy of the player *i* and –*i*
  - Players can be homogenous or heterogenous, symmetric or asymmetric
  - Most of our games we will consider two player (n=2)

Information

- Information governs how players choose their strategies, how they perceive their utility and payoffs and eventually the outcome of the game

- Games can have perfect, complete, or incomplete information

- For simplicity we will focus on complete information for the time being.

Types of games

- Single vs repetitive
- Simultaneous vs sequential
- Deterministic vs Stochastic
- Cooperative and non-cooperative

Types of Games (Examples)

- Coin flip
- Rock, Paper, Scissors
- Prisoners’ dilemma
- Bidding
- Claims
- Traffic behavior examples

Strategies, Strategy profiles and Equilibriums

Strategy profiles

- A strategy (s_i) is how a player (*i) will choose an* action against other players.

- A strategy profile is a mix of strategies between players

$$(s_1, s_2, \ldots, s_i, s_{i+1}, \ldots, s_n)$$

$$(s_i, s_{-i})$$

How to choose a Strategy?

Equilibrium

- An Equilibrium is a strategy profile s* that is made of the best strategy of each player *i.*

$$s^* = (s_i^*, s_{-i}^*)$$

- Equilibrium strategy vs equilibrium outcome

Equilibrium (solution) Concept

- We need to define the "best strategy" and what does that mean for the game.

- An Equilibrium Concept = $F: \{S_1, S_{-i,} \pi_1, \pi_{-i}\} \rightarrow s^*$
  - i.e.; a function or rule that takes all strategy profiles with their payoffs for the players and define the equilibrium $s*$

Dominate Strategies

- A dominate strategy is a strategy in set S_i that no other strategy can provide better outcome for player *i*.

$$\boldsymbol{\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}) \qquad \forall s_{-i}, \forall\ s_i^* \neq s_i'}$$

Dominated strategy(s_i^d): $\exists s_i', such\ that, \pi_i\left(s_i^d, s_{-i}\right) < \pi_i(s_i', s_{-i})$

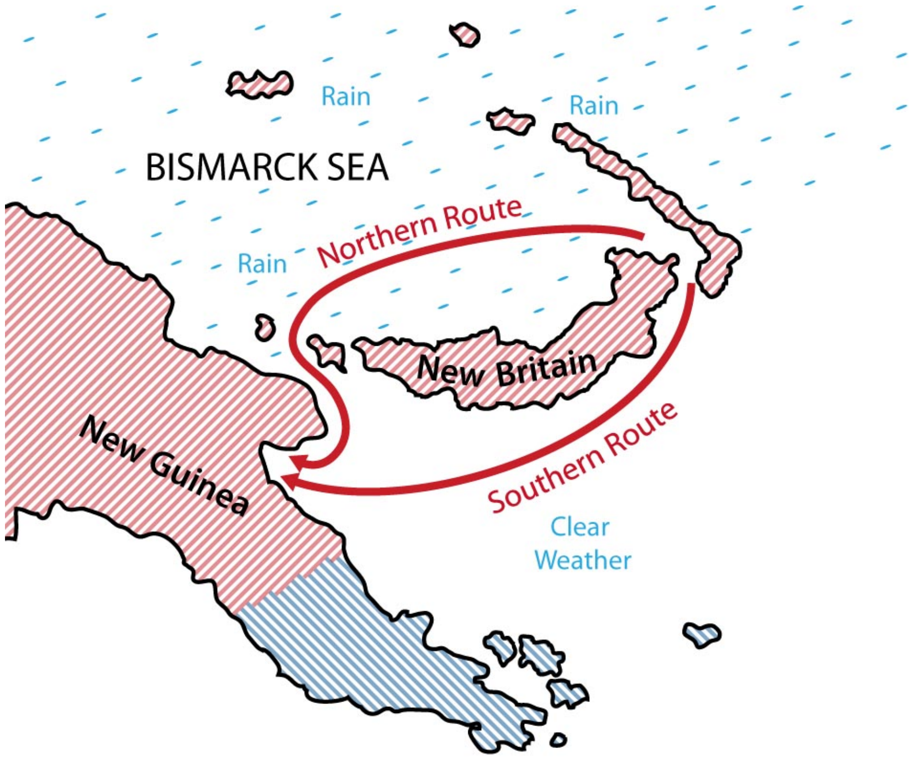
- Dominate strategy equilibrium is the strategy profile made of each player's dominate strategy for the game.

- Prisoner dilemma (practice)

Not all games have a dominate strategy

- South Pacific Naval Battel (1943)

		General Imamura	
General Kenny		North	South
	North	2,-2	2,-2
	South	1,-1	3,-3



Iterated Dominance

- A strategy $s_i'$is ***weakly dominated*** if there exists a strategy s_i'' that is as good as it for playing other players, but s_i' is worse in at least one case.

$$\boldsymbol{\pi_i(s_i'', s_{-i}) \geq \pi_i(s_i', s_{-i}) \qquad \forall s_{-i},}$$
$$\boldsymbol{and}$$
$$\boldsymbol{\exists s_{-i}\ such\ that, \pi_i(s_i'', s_{-i}) > \pi_i(s_i', s_{-i})}$$

A ***Weakly dominate*** strategy will produce "***weak-dominance equilibrium***" by removing the weakly strategies in an iterated manner.

Another example

Battel of the sexes

Bob

Alice		Ballet	Soccer
	Ballet	2,1	0,0
	Soccer	0,0	1,2

Iteration Path Game

		Column		
		C1	C2	C3
Row	R1	2, 12	1,10	1,12
	R2	0,12	0,10	0,11
	R3	0,12	1,10	0,13

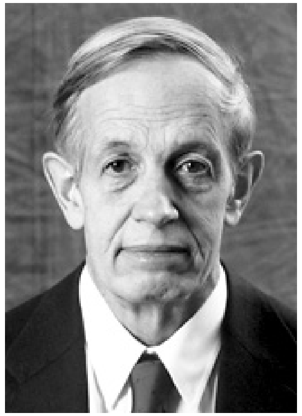
Nash Equilibrium

- Nash equilibrium is the most recognized stable strategy profile.

- A strategy profile is a Nash Equilibrium (NE) if no player has an incentive to deviate from his/her strategy, given no one will deviate from their strategies as well.

$$\forall i, \quad \pi_i \quad (s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \qquad \forall s_i'$$

- NE is the best response of the players to each others.



John Nash (1928 – 2015)

Construction Research Example

		Small Company	
		Research	Don't Research
Big Company	Research	5,1	4,4
	Don't Research	9,-1	0,0

$$\forall i, \quad \pi_i \quad (s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \qquad \forall s_i'$$

Note, there is no $\forall s_{-i}'$

NE. is the best response to the others' best strategies

Weak, Strong N.E., and Preto-Efficient

- Prisoners' Dilemma

		Player2	
		Deny	Confess
Player 1	Deny	-1,-1	-10,0
	Confess	0,-10	-8,-8

- Modeller's Dilemma

		Player2	
		Deny	Confess
Player 1	Deny	0,0	-10,0
	Confess	0,-10	-8,-8

Another example

Battel of the sexes

		Bob	
		Ballet	Soccer
Alice	Ballet	2,1	0,0
	Soccer	0,0	1,2

Iteration Path Game

		Column		
		C1	C2	C3
Row	R1	2, 12	1,10	1,12
	R2	0,12	0,10	0,11
	R3	0,12	1,10	0,13

Construction Developers Game

- Let's consider two construction developers constructing new compounds in Cairo. Each can produce a quantity of *q* per year.
- Lets consider a marginal cost of *c* per *q*
- price per constructed unit = $P(Q) = M – Q$, where $Q = \sum q$, M is the available market value

- How can we find the Nash Equilibrium?
  - Lets consider both monopolist, and collusive behaviors

Mixed Strategies

- We used Nash Equilibrium (N.E.) to find equilibrium strategy profiles for games with no Dominate Strategy.
- In some games, there is even no N.E.
- We need to expand the strategy space to include random actions
- These strategies are defined as Mixed Strategies

Mixed Strategies

- Pure Strategy
  - A strategy made of a single action a player *i* can take

$$s_i\colon\ \omega_i \to a_i$$

- Mixed Strategy
  - Probability distribution over actions that allows the player *i* to chose the different actions

$$s_i\colon\ \omega_i \to m(a_i), \quad where, m \geq 0,\ and \quad \int_{A_i} m(a_i)da_i = 1$$

NOTE:

If m of an action is = 0, then it will never be chosen

if m = 1, then it's a pure strategy

Either way, it will be in a pure strategy setting

Mixed Strategy

- If pure strategy points out which exact action a player should take, a mixed strategy tells the player with what probability should he/she plays the action.

- This unpredictability gives the player(s) an edge and useful to all players.

Welfare Game

- The government has two actions, to aid a citizen in need, or not.
- The citizen can either try to find a job, or enjoys a free ride
- The government would rather help a hard working citizen than a free-rider.

`		Citizen	
		Seek Job (ɣ)	Do not (1-ɣ)
Government	Aid (ɸ)	3,2	-1,3
	Not Aid (1-ɸ)	-1,1	0,0

- There is no Dominate Strategy, No N.E.

Welfare Game

- Solving it as a maximization problem
- $\gamma^* = 0.2$
- $\phi^* = 0.5$
- MSNE: $s^* = (\phi^*, \gamma^*)$

		Citizen	
		Seek Job (γ)	Do not ($1-\gamma$)
Government	Aid (ϕ)	3,2	-1,3
	Not Aid ($1-\phi$)	-1,1	0,0

- We found the probability of citizens utilizing the actions that makes the government indifferent among its actions

Payoff-Equating Method for Mixed Strategy

- Payoff-Equating method is a simpler approach when you know which strategies you will mix.

- Simply, for the same player, the payoff from each pure strategy should be the same in mixed strategy

$$\pi_G(Aid) = \pi_G(Do\ not\ Aid)$$

$$\theta\gamma \times 3 + \theta(1-\gamma) \times -1 = (1-\theta)(\gamma) \times -1$$

Dynamic Games

- Dynamic or sequential games are common in various engineering situations
  - Traffic
  - Bidding
  - Claims

Dynamic games

- Nodes
- Successors
- Predecessor
- Edges
- End-nodes