Cooperative Game Theory

Sharing the benefits of collaboration

(Transferable Utility Games)

CB714 – Advanced System Analysis

Mohamed S. Eid, PhD

Introduction

- Alice, Bob and Carol want to by ice cream.

- Three different size of ice cream buckets (500gm, 750gm, 1000gm)

- Alice has $6
- Bob has $4
- Carol has $3

Introduction

- None can buy any by him/herself
- They need to collaborate to buy at least the 500gm bucket.
- How should they share it?

Cooperative Game Theory

- How to define a stable share among the participants.

- In engineering
  - Construction
  - Water resources
  - Traffic engineering

Cooperative Game Theory

- Formal notation
- Game G(*N,v*)
  - N is the number of players
  - V is the characteristic function of the coalition (its worth)

- Coalition S⊆N
- $\mathrm{If\ S = 0, then\ the\ coalition\ is\ empty\ (no\ players)}$
- $\mathrm{If\ S = 1\ its\ called\ a\ singleton}$
- $\mathrm{The\ grand\ coalition\ is\ when\ S = N}$

Cooperative Game Theory

- Alice (a), Bob (b) and Carol (c)

S = 0, $S = \{\emptyset\}$;
$S = 1$ when $S = \{\{a\},\{b\},\{c\}\}$;
$S = 2$, when $S = \{\{a,b\},\{a,c\},\{b,c\}\}$;
and $S = N = 3$, when $S = \{a,b,c\}$,

Cooperative Game Theory

- Each coalition has its worth, what they can achieve (v)

S = 0

v(0) = 0

$S = 1$

v(a) = 0, v(b) =0, v(c)= 0;

$S = 2,$

v(a,b) = 750, v(a,c) = 750, v(b,c) = 500

and $S = N = 3,$

v(a,b,c) = 1000

Formally, $S \subseteq N \rightarrow v(S)$.

- Alice has \$6
- Bob has \$4
- Carol has \$3

Cooperative Game Theory

- Supper-additivity of games
  - When the worth of the coalition is greater than the sum of its parts

$$v(S+i) \geq v(S) + v(i), i \notin S$$

Sharing the Coalition Worth

- The share of each player of the coalition's collective output is denoted as $x_i,\ i \in S$.

- Thus, the output of a game (N, v) is a pair of $(S, x)$where $S = (1, 2, 3, \ldots, N)$ of different coalition structures, and $x = (x_i\ \forall\ i \in S)$ is a payoff vector for the distribution of the value of the coalition.

- The share must remain rational
  - $x_i \geq v(i), \forall i \in S$

The Core

- The Core determines a set of possible solutions for a stable coalition.

- If players in coalition S^* prefer their shares at coalition S, then it is said that S^* $dominates$ S.

$$\forall\, i \;\in S, x_{i(s)} \geq x_{i(s')}, where\ S' \neq S$$

$$\forall\, S \;\subseteq N, \sum_{i \in S} x_i \geq v(S)$$

The Core – Glove Game

- Alice and Bob produce left gloves, while Carol produces right gloves
- $(a) \ = \ v(b) \ = \ v(c) \ = \ 0.$
- $v(a, b) \ = 0, \ v(a, c) \ = \ 1, v(b, c) \ = \ 1$
- The grand coalition would also provide a value of unity; $v(a, b, c) \ = 1.$
- **Case #1**: $x_a \ = 0.25$; x_b= 0.25; $x_c \ = 0.5$
- This share is *not* in the Core. Not stable at S = 2
- **Case #2**: $x_a \ = 0.0$; x_b= 0.0; $x_c \ = 1$

$$\forall \ i \ \in S, x_{i(s)} \geq x_{i(s')}, where \ S' \neq S$$

$$\forall \ S \ \subseteq N, \sum_{i \in S} x_i \geq v(S)$$

The Core – Glove Game

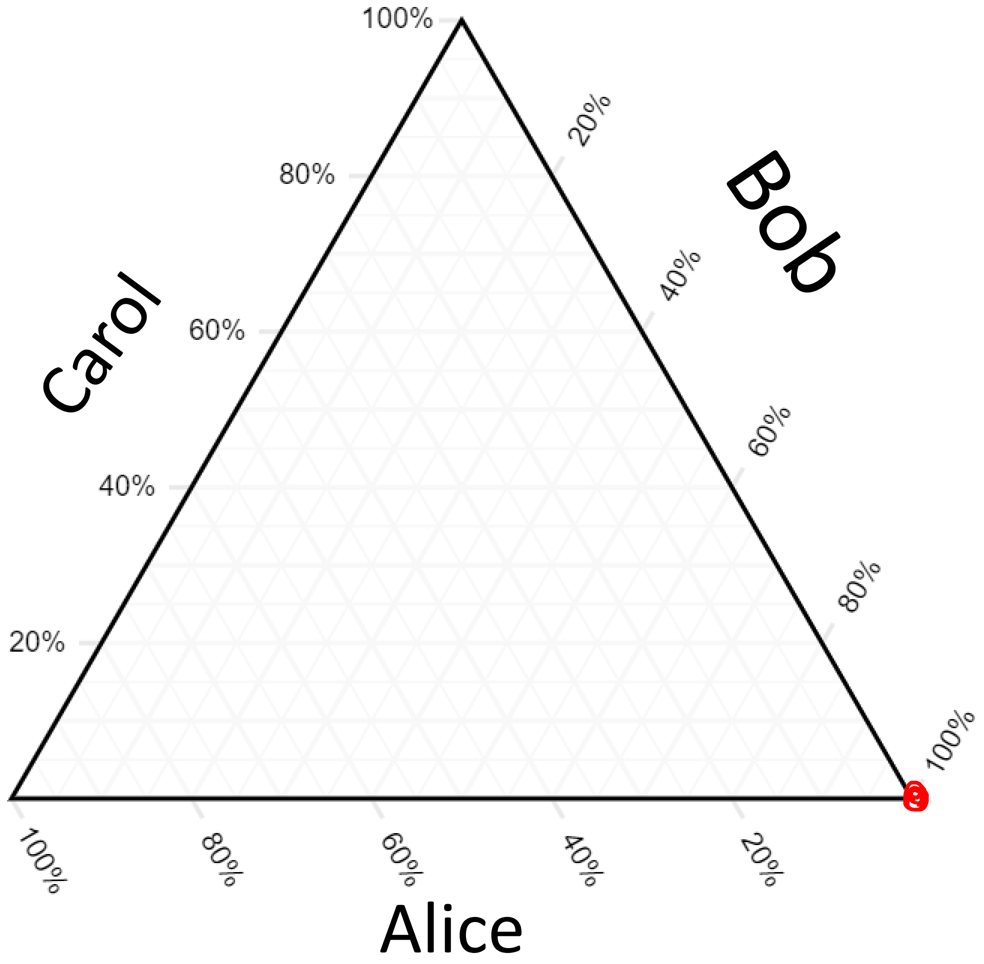
- Ternary plot

- $x_a, x_b, x_c \geq 0$
- $x_a + x_b + x_c = 1$

- $x_a + x_c \geq 1$, therefore, $x_b \leq 0$

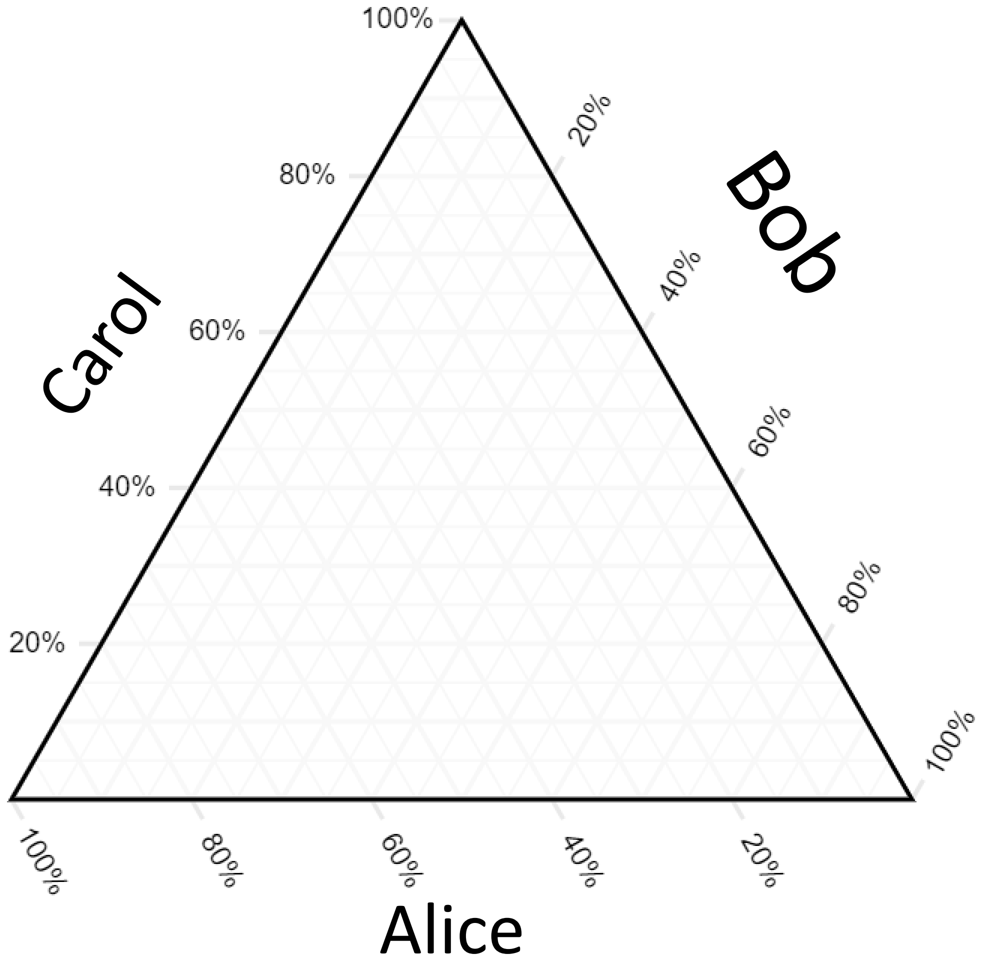
- $x_b + x_c \geq 1$, therefore, $x_a \leq 0$

- $x_a + x_b \geq 0$, therefore, $x_c \leq 1$



Ice Cream Game – The Core

- $since, x_a + x_b + \ x_c = 1000$
- $and\ since, x_a + \ x_b = 750, then\ x_c \leq 250$
- $and\ since, x_a + \ x_c = 750, then\ x_b \leq 250$
- $and\ since, x_b + \ x_c = 750, then\ x_a \leq 500$



Shapley value

Fair Shares

Shapley value (Shapley 1953)

1) **Efficiency:** the sum of the shares of each player should be equal to the value of the coalition

2) **Symmetry**: if two players provide the same outcome, both should get the same share

3) **Dummy player:** a player that does not contribute to the coalition should get the share of him/herself being alone.

4) **Additivity:** for two different coalitional games involving the same set of individuals, if the setting is restructured as a single coalition, the individual's payments should equal the sum of payments that they would have achieved in two separate games.

Shapley Value

- Marginal Contribution of a player to a coalition
  - How much to I impact/benefit the coalition

Marginal contribution of player i in coalition S $[v(S) - v(S - \{i\})]$

- Share should be proportional to the contribution

Shapley Value

$$\varphi_i = \sum_{S \subseteq N_{i \in S}} \frac{(|S| - 1)!\,(|N| - |S|)!}{N!} \,.\, [v(S) - v(S - \{i\})]$$

Shapley value for ice cream game

- Each coalition has its worth, what they can achieve (v)

S = 0

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$S = 1$

v(a) = 0, v(b) =0, v(c)= 0;

$S = 2$,

v(a,b) = 750, v(a,c) = 750, v(b,c) = 500

and $S = N = 3$,

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Formally, $S \subseteq N \rightarrow v(S)$.

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Shapley value for ice cream game

$$\varphi_i = \sum_{S\subseteq N_{i\,\in S}} \frac{(|S|-1)!\,(|N|-|S|)!}{N!}\,.\,[v(S)-v(S-\{i\})]$$

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$S = 1$

v(a) = 0, v(b) =0, v(c)= 0;

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Glove Game

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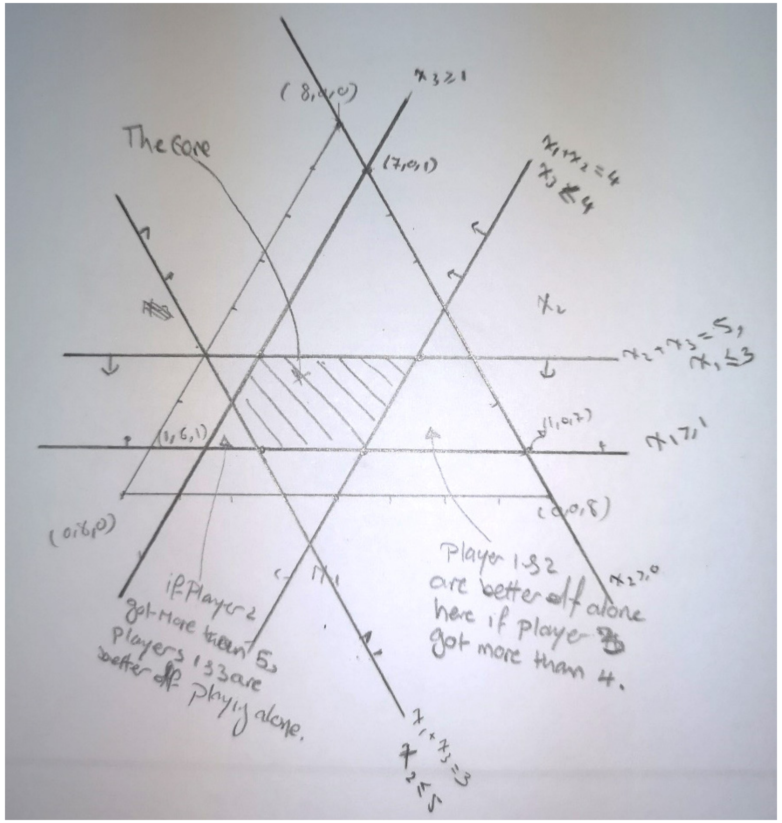
- The Core:

$x_a = 0.0$; x_b= 0.0; $x_c = 1$

- $v(1) \geq 1$
- $v(2) \geq 0$
- $v(3) \geq 1$
- $v(1,2) \geq 4$
- $v(1,3) \geq 3$
- $v(2,3) \geq 5$
- $v(1,2,3) = 8$



Propensity to disrupt

- Gately (1974)

- Accounting for the negotiation power of each player

$$PTDi(x) = \frac{\sum_{j \neq i} x(j) - v(N \setminus \{i\})}{x(i) - v(\{i\})}$$

Other solutions

- Kernel
- Nucleolus
- Owen